THE CROSSING NUMBER AND MAXIMAL BRIDGE LENGTH OF A KNOT DIAGRAM

This is a preprint. I would be grateful for any comments and corrections!

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Current version: February 1, 2008 First version: February 15, 1999

Abstract. We give examples showing that Kidwell's inequality for the maximal degree of the Brandt-Lickorish-Millett-Ho polynomial is in general not sharp.
Keywords: Brandt-Lickorish-Millett-Ho polynomial, unknotting number, crossing number, bridge length
AMS subject classification: 57M25

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1. Introduction

The Q (or absolute) polynomial is a polynomial invariant in one variable z of links (and in particular knots) in S^3 without orientation defined by being 1 on the unknot and the relation

$$A_1 + A_{-1} = z(A_0 + A_\infty)\,, \tag{1}$$

where A_i are the Q polynomials of links K_i and K_i ($i \in \mathbb{Z} \cup \{\infty\}$) possess diagrams equal except in one room, where an i-tangle (in the Conway [Co] sense) is inserted, see figure 1.

It has been discovered in 1985 independently by Brandt, Lickorish and Millett [BLM] and Ho [Ho]. Several months after its discovery, Kauffman [Ka] found a 2-variable polynomial $F(a,z)$, specializing to Q by setting $a = 1$.

In [Ki], Kidwell found a nice inequality for the maximal degree of the Q polynomial.

*Supported by a DFG postdoc grant.

†All my papers, including those referenced here, are available on my webpage or by sending me an inquiry to the email address I specified above. Hence, *please*, do not complain about some paper being non-available before trying these two options. Thank you.

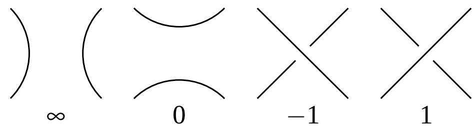


Figure 1: The Conway tangles.

Theorem 1.1 (Kidwell) Let D be a diagram of a knot (or link) K. Then

$$\max\deg Q(K) \leq c(D) - d(D)\,, \tag{2}$$

where $c(D)$ is the crossing number of D and $d(D)$ its maximal bridge length, i. e., the maximal number of consecutive crossing over- or underpasses. Moreover, if D is alternating (i. e. $d(D) = 1$) and prime, then equality holds in (2).

In [Mo, problem 4, p. 560] he asked whether (2) always becomes equality when minimizing the r.h.s. over all diagrams D of K. From the theorem it follows that this is true for alternating knots and also for those non-alternating knots K, where $\max\deg Q(K) = c(K) - 2$ (here $c(K)$ denotes the crossing number of K). All non-alternating knots in Rolfsen's tables [Ro] have this property except for one – the Perko knot 10_{161} (and its obversed duplication 10_{162}), where $\max\deg Q = 6$. Hence, as quoted by Kidwell, this knot became a promising candidate for strict inequality in (2). To express ourselves more easily, we define

Definition 1.1 Call a knot K Q-maximal, if (2) with the r.h.s. minimized over all diagrams D of K becomes equality.

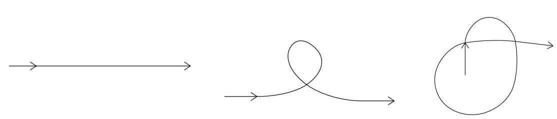
The aim of this note is to show that indeed the Perko knot is not Q-maximal. We give several modifications of our arguments and examples showing how they can be applied to exhibit non-Q-maximality.

2. Plane curves

We start by some discussion on plane curves.

Definition 2.1 A non-closed plane curve is a C^1 map $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(0) \neq \gamma(1)$ and only transverse self-intersections. γ carries a natural orientation.

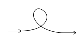
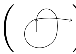
Example 2.1 Here are some plane curves:



In the following, whenever talking of plane curves we mean non-closed ones with orientation unless otherwise stated. However, in some cases it is possible to forget about orientation if it is irrelevant. It is convenient to identify γ with $\gamma([0,1])$ wherever this causes no confusion. Whenever we want to emphasize that a line segment in a local picture starts with an endpoint, the endpoint will be depicted as a thickened dot.

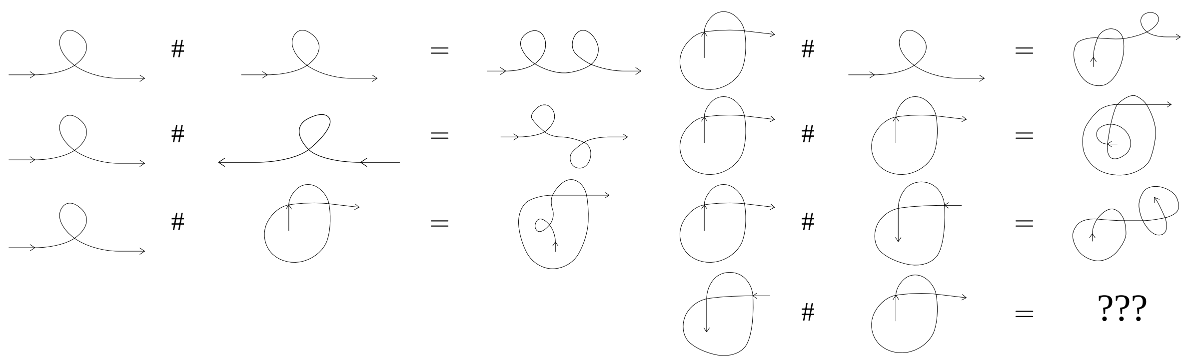
Definition 2.2 The crossing number $c(\gamma)$ of a curve γ is the number of self-intersections (crossings). The curve γ with $c(\gamma) = 0$ is called trivial.

Definition 2.3 We call a non-closed curve $\tilde\gamma$ similar to γ (and denote it by $\tilde\gamma \sim \gamma$) if $\tilde\gamma(0) = \gamma(0)$, $\tilde\gamma(1) = \gamma(1)$ and $\tilde\gamma$ intersects γ only transversely. The distance $d(\gamma)$ of γ call the number $\min\{\,\#(\gamma\cap\tilde\gamma) : \tilde\gamma \sim \gamma\} - 2$ (the '-2' provided to ignore the coincidence of start- end endpoint). A curve $\tilde\gamma$ realizing this minimum is called minimal similar to γ. Such $\tilde\gamma$ can be chosen to have no self-intersections.

Example 2.2 The curves and  have $d = 0$, while $d\Big($  $\Big) = 1$.

Definition 2.4 We call a plane curve γ composite, if there is a *closed* plane curve γ' (with no self-intersections) such that γ' intersects γ in exactly one point, transversely, and in both components of $\mathbb{R}^2 \setminus \gamma'$ there are crossings of γ. In this case γ' separates γ into two parts γ_1 and γ_2, which we call components of γ. We write $\gamma = \gamma_1 \# \gamma_2$. Conversely, this can be used to define the operation '#' (connected sum) of γ_1 and γ_2, wherever $\gamma_1(1)$ or $\gamma_2(0)$ are in the closure of the unbounded component of their complements. We call γ prime, if it is not composite.

Example 2.3



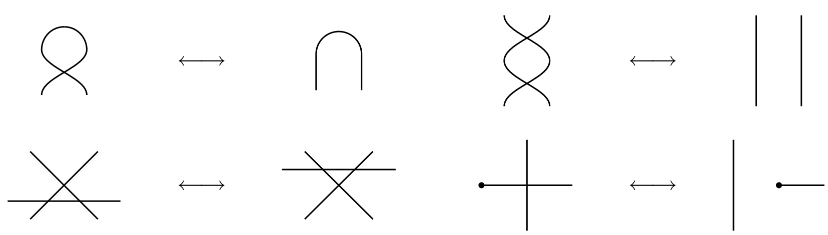
This example visualizes that the connected sum in general depends on the orientation of the summands and their order.

It is clear that the crossing number is additive under connected sum and it's a little exercise to verify that the distance is as well.

The path we are going to follow starts with the following

Exercise 2.1 Verify that the complement of a curve γ has $c(\gamma) + 1$ connected components, and conclude from that that $d(\gamma) \le c(\gamma)$.

Hint: One way to show that is to observe it when γ is trivial, to prove that you can obtain any γ from the trivial one by the four local moves



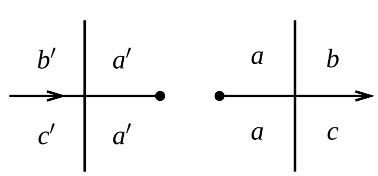
(3)

and to trace how the number of components and $c(\gamma)$ change under these moves.

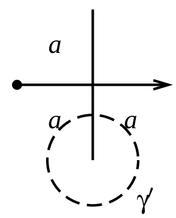
Our first aim is to improve slightly the inequality of the exercise.

Lemma 2.1 $d(\gamma) \le \max(2, c(\gamma) - 2)$ if γ prime.

Proof. Consider the first (and analogously last) crossing of γ (that is, the crossings passed as first and last by γ). Denote by letters the connected components of the complement near these crossings:

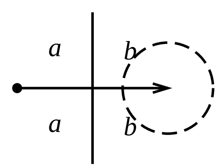


First note that $a \notin \{b,c\}$, else if w.l.o.g. $a = c$ there would be a closed curve γ' like



intersecting γ in only one point and either $d(\gamma) = 0$ or γ is composite.

Then note that $b \neq c$, because else there would be a γ' like



and the 2 curve segments could not be connected.

Therefore, $b \neq c$ and a minimal curve $\tilde{\gamma} \sim \gamma$ would not need to pass through one of b and c. The same holds for the last crossing of γ. Hence we avoid $\tilde{\gamma}$ passing through at least two components of the complement of γ, unless $\{b,c\} \cap \{b',c'\} \neq \varnothing$, but then $d(\gamma) \leq 2$. □

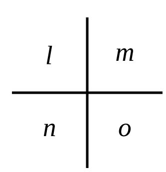
For the Perko knot we need to work a little harder.

Lemma 2.2 $d(\gamma) \leq \max(3, c(\gamma) - 3)$ if γ prime.

Definition 2.5 An isolated crossing of γ is a crossing p such that there is a closed curve γ' with $\gamma \cap \gamma' = \{p\}$ and γ' intersects transversely both strands of γ intersecting at p.

Proof of lemma. If γ has an isolated crossing, then one of the components of $\mathbb{R}^2 \setminus \gamma$ has both and the other one has no one of the endpoints of γ. Removing the part of γ in latter component and smoothing γ near p reduces $c(\gamma)$, but not $d(\gamma)$, hence we may (say, by induction on $c(\gamma)$) assume that γ has no isolated crossing.

Now consider a crossing of γ which is neither the first nor the last and denote the components near it by l, m, n and o.



(4)

Call 2 components neighbored if the intersection of the closures of their fragments in (4) is a line, and opposite if it is just the crossing itself.

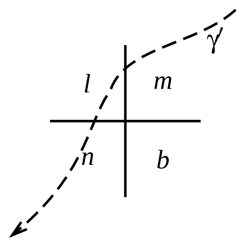
By primality of γ any two neighbored components are distinct and by non-isolatedness of the crossing so are any two opposite components.

Hence l, m, n and o are pairwise distinct. Now call b, b' the components which were found not to be passed by a minimal similar curve γ' to γ by the proof of lemma 2.1 and a, a' the components denoted so in the same proof.

Then distinguish some cases.

Case 1. No one of b, b' is among l, m, n and o. As γ' certainly does not pass through one of l, m, n and o you have a third component not passed by γ' and you are done.

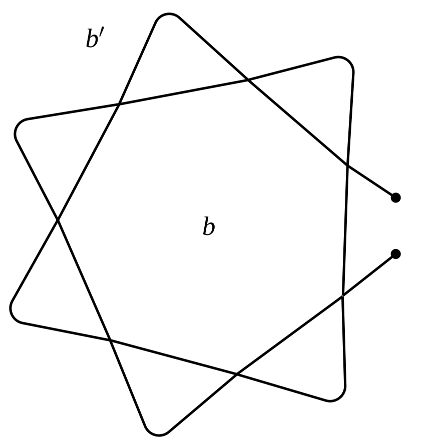
Case 2. Exactly one of b, b', say b, is among l, m, n and o. You would be done as in case 1 unless γ' does not pass only through b. Then you have a picture like this:



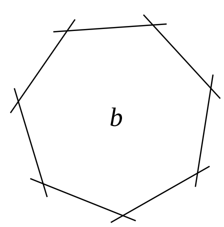
Then γ' passes through m and n, w.l.o.g. first through m and then through n. But then γ' is not minimal because passing through a, c and m (and possible further components between c and m) before passing through n could be replaced by just passing through a and b to arrive to n. By this contradiction you are done here.

Case 3. Both b, b' are among l, m, n and o. If b and b' are neighbored, then $d(\gamma) \leq 3$.

Therefore, by this case distinction you are done unless at any crossing of γ except the first and the last one b and b' participate as opposite components. In particular b participates as a neighboring component at any crossing of γ except possibly the last one. But then one can see that γ must look like



To see this, start with



and then observe that there is only one way to reconnect the outer arcs not creating crossings (except possibly the last one) and having b' as specified, and moreover it works only if the number of crossings is odd.

But for such a curve $d(\gamma) = 0$. □

3. Non-Q-maximal knots

Now we are prepared to exhibit the Perko knot as non-Q-maximal.

Theorem 3.1 If D is a prime diagram of a knot K of $c(D)$ crossings with a bridge of length $l = c(D) - k$, and D has minimal crossing number among all such diagrams for fixed k, then $l \leq \max(3, k-3)$, hence $c(D) \leq k + \max(3, k-3)$.

From this we have the desired example:

Example 3.1 If 10_{161} were Q-maximal, then we could pose $k = 6$ in the theorem and would obtain a 9 crossing diagram of the knot, which does not exist. Hence 10_{161} is not Q-maximal.

Proof of theorem. This is basically lemma 2.2. Consider γ' to be the part of D consisting of the maximal (length) bridge and γ consisting of the rest of (the solid line of) D with signs of all crossings ignored. Then the freedom to move the bridge corresponds to the freedom to move γ'. $\square$

Clearly, for many phenomena Rolfsen's tables up to 10 crossings are very limited. Verifying the list of non-alternating knots of at most 15 crossings provided by Thistlethwaite (see [HTW]), I found 189 15 crossing knots for which $\max\deg Q \le 8$, and hence for which we would be done showing non-Q-maximality already with lemma 2.1 (or even exercise 2.1). The most striking examples are the knots 15_{119574} and 15_{119873}, where $\max\deg Q = 4$ (although for both $\max\deg_z F(a,z) = 11$, the coefficients of the 7 highest powers of z cancel when setting $a = 1$).

There are several ways how the theorem can be modified.

Theorem 3.2 If D is a diagram of a knot K of $c(D)$ crossings with a bridge of length $l = c(D) - k$, then $u(K) \le \lfloor k/2 \rfloor$, where $u(K)$ denotes the unknotting number of K.

Proof. By switching at most half of the crossings in D not involved in the maximal bridge, the remaining part γ of the plane curve (this time *with* signs of the crossings) can be layered, i. e., any crossing is passed the first time as over- and then as under-crossing or vice versa. But reinstalling the bridge to a layered γ gives a layered, and hence unknotted, diagram. $\square$

Corollary 3.1 If $u(K) > \lfloor \max\deg Q(K)/2 \rfloor$, then K is not Q-maximal. $\square$

Unfortunately, this corollary does not work to show non-Q-maximality of Perko's knot. Verifying both hand-sides of the inequality (using that the unknotting number of 10_{161} is 3, see [St, Km, Ta]), we find that we just have equality. And that equality does not suffice is seen, e. g., from all 8 closed positive braid knots in Rolfsen's tables (see [Cr, Bu]) and more generally from the $(2,n)$-torus knots for n odd.

For knots of > 10 crossings unknotting numbers are not tabulated (anywhere I know of) and a general machinery does not exist to compute them, hence when wanting to extend the search space for examples applicable to corollary 3.1, it makes sense to replace the unknotting number by lower bounds for it, which can be computed straightforwardly. I tried two such bounds. First we have the signature σ.

Corollary 3.2 If $|\sigma(K)| > \max\deg Q(K)$, then K is not Q-maximal. $\square$

Clearly, replacing Q by lower bounds for it makes the condition more and more restrictive. However, when checking the above mentioned list of 189 knots, I found that at least one of them satisfied strict inequality. It is 15_{166028}, where $\sigma = 8$ and $\max\deg Q = 7$.

Another possibility is to minorate $u(K)$ by the bound coming from the Q polynomial itself.

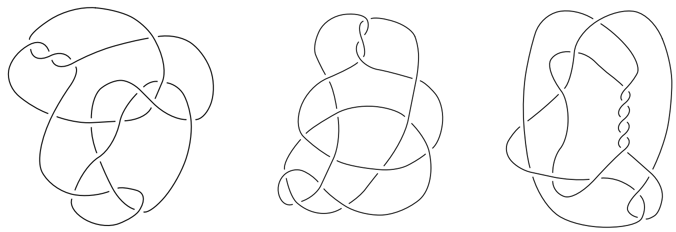
Corollary 3.3 If $2\log_{-3} Q(-1) > \max\deg Q(K)$, then K is not Q-maximal. $\square$

Remark 3.1 The negative logarithm base may disturb the reader because such logarithms are usually not defined. But by work of Sakuma, Murakami, Nakanishi (see Theorem 8.4.8 (2) of [Kw]) and Lickorish and Millett [LM] $Q(-1)$ is always a(n integral) power of -3 and this one it is referred to by this expression.

The inequality in corollary 3.3 looks rather bizarre. First, the inequality $u(K) \ge \log_{-3} Q(-1)$ is in general much less sharp than the one with the signature and secondly, the inequality in corollary 3.3 requires the coefficients of Q to be of an average magnitude which grows exponentially with $\max\deg Q$. Thus, non-surprisingly, my quest for applicable examples among the non-alternating 15 and 16 crossing knots ended with no success in this case.

Question 3.1 Is there a knot K with $2\log_{-3} Q(-1) > \max\deg Q(K)$?

I nevertheless gave the above inequality, because it is self-contained w.r.t. Q and would decide about non-Q-maximality from Q itself (without knowing anything else about the knot) and hence is, in some sense, also beautiful.



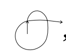
15_{119574} 15_{119873} 15_{166028}

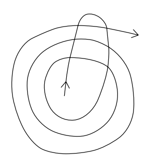
Figure 2: Three non-Q-maximal knots.

4. A question on plane curves

The machinery of the dependence of $d(\gamma)$ on $c(\gamma)$ we developed just as far as necessary for our knot theoretical context, but possibly it is also interesting in its own right.

Question 4.1 Which is the best upper bound for $d(\gamma)$ in terms of $c(\gamma)$, i. e. a function $f:\mathbb{N}\to\mathbb{N}$ such that for any γ we have $f(c(\gamma))\geq d(\gamma)$?

We proved that for *prime* γ we can choose $f(n):=\max(3,n-3)$. Turning back to our example , where $c=2$ and $d=1$, and applying connected sums we find that we cannot choose $f(n)$ better than $\lfloor n/2\rfloor$ (similar prime examples as



exist as well). But possibly this indeed is the best upper bound. Unfortunately, proving it seems a matter of further tricky labour as in §2.

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